APPLIED MATHEMATICS 1

(CBCGS 2016)

Q1]a) If $\cos\alpha\cos\beta = \frac{x}{2}$, $\sin\alpha\sin\beta = \frac{y}{2}$ ***, prove that :-*** **(3)**

$$\sec(\alpha - i\beta) + \sec(\alpha - i\beta) = \frac{4x}{x^2-y^2}$$

Solution:- $\cos\alpha\cos\beta = \frac{x}{2}$ *and* $\sin\alpha\sin\beta = \frac{y}{2}$ ………………*(given)*

$$\sec(\alpha - i\beta) = \frac{1}{\cos(\alpha-i\beta)} = \frac{\frac{1}{\cos\alpha\cos i\beta+\sin\alpha\sin i\beta}}{\cos\alpha\cosh\beta+i\sin\alpha\sinh\beta} \frac{1}{\frac{x}{2}+\frac{iy}{2}} = \frac{2}{x+iy} \quad \text{……………………(1)}$$

$$\sec(\alpha + i\beta) = \frac{2}{x-iy} \quad \text{………………(2)}$$

from (1) and (2)

$$\sec(\alpha - i\beta) + \sec(\alpha - i\beta - i\beta = \frac{2}{x+iy} + \frac{2}{x-iy} = \frac{4x}{x^2-y^2}$$

Q1]b) If $Z = \log(e^x + e^y)$ ***show that*** $rt - s^2 = 0$ ***where*** $r = \frac{\partial^2 Z}{\partial x^2}$, $t = \frac{\partial^2 Z}{\partial y^2}$ $s = \frac{\partial^2 Z}{\partial x\,\partial y}$

Solution :- **(3)**

$Z = \log(e^x + e^y)$

(1) $\frac{\partial z}{\partial x} = \frac{e^x}{(e^x+e^y)}$ $\quad \frac{\partial^2 Z}{\partial x^2} = \frac{e^x(e^x+e^y)-e^x(e^x)}{(e^x+e^y)^2} = \frac{e^{2x}+e^{xy}-e^{2x}}{(e^x+e^y)^2}$

$$r = \frac{\partial^2 Z}{\partial x^2} = \frac{e^{xy}}{(e^x+e^y)^2} \quad \text{…………(1)}$$

(2) $\frac{\partial z}{\partial y} = \frac{e^y}{(e^x+e^y)}$ $\frac{\partial^2 Z}{\partial y^2} = \frac{e^y(e^x+e^y)-e^y(e^y)}{(e^x+e^y)^2} = \frac{e^{2y}+e^{xy}-e^{2y}}{(e^x+e^y)^2}$

$t = \frac{\partial^2 Z}{\partial y^2} = \frac{e^{xy}}{(e^x+e^y)^2}$ ....(2)

(3) $\frac{\partial z}{\partial x} = \frac{e^x}{(e^x+e^y)}$ $s = \frac{\partial^2 Z}{\partial x\,\partial y} = \frac{e^{xy}}{(e^x+e^y)^2}$ ....(3)

From (1) , (2) and (3) we get ,

$rt = \left(\frac{e^{xy}}{(e^x+e^y)^2}\right)\times\left(\frac{e^{xy}}{(e^x+e^y)^2}\right) = \left(\frac{e^{xy}}{(e^x+e^y)^2}\right)^2 = \left(\frac{e^{2xy}}{(e^x+e^y)^2}\right)$ ....(4)

$s^2 = \left(\frac{e^{xy}}{(e^x+e^y)^2}\right)^2 = \left(\frac{e^{2xy}}{(e^x+e^y)^2}\right)$ ....(5)

From (4) and (5) we get,

$rt - s^2 = \left(\frac{e^{2xy}}{(e^x+e^y)^2}\right) - \left(\frac{e^{2xy}}{(e^x+e^y)^2}\right) = 0.$

Hence proved $rt - s^2$ *= 0*

Q1] c) If x=uv , y = $\frac{u+v}{u-v}$ ***. find*** $\frac{\partial(u,v)}{\partial(x,y)}$. ***(3)***

Solution:- $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

x = uv, $y = \frac{u+v}{u-v}$ *...........(given)*

we know that JJ' = 1 ....(1)

the equation can also be solved by this following method.

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$\frac{\partial x}{\partial u} = \partial(uv) = v.$ *............... (2)*

$\frac{\partial x}{\partial v} = \partial(uv) = u.$ *...............(3)*

$$\frac{\partial y}{\partial u} = \partial\left(\frac{u+v}{u-v}\right) = \frac{(u-v)-(u+v)}{(u-v)^2} = \frac{-2v}{(u-v)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\frac{\partial y}{\partial v} = \partial\left(\frac{u+v}{u-v}\right) = \frac{(u-v)+(u+v)}{(u-v)^2} = \frac{2u}{(u-v)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

From equation (2), (3), (4), (5) we get,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix} = \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}.$$

From (1) we get,

JJ' = 1

$$J \times \frac{4uv}{(u-v)^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots(\text{let } J' = \frac{4uv}{(u-v)^2})$$

Hence $J = \frac{(u-v)^2}{4uv}.$

$$\therefore e^{2\varphi} = \cot\frac{\alpha}{2}$$

Q1] d) If $y = 2^x \sin^2 x \cos x$ ***find*** y_n **(3)**

Solution :- $2^x = e^{x\log 2} = e^{ax}$ where $a = \log 2$

$$\frac{2\sin^2 x\cos x}{2} = \frac{\sin^1 x\cos x.\sin x \times 2}{2} = \frac{\sin x.\sin 2x}{2} = \frac{2\sin x.\sin 2x}{2\times 2} = \frac{\cos x}{4} - \frac{\cos 3x}{4}$$

$$\therefore \sin^2 x\cos x = \frac{\cos x}{4} - \frac{\cos 3x}{4}$$

$$Y = \frac{e^{ax}\cos x}{4} - \frac{e^{ax}\cos 3x}{4}$$

$$y_n = \tfrac{1}{4} r_1^n e^{ax} \cos\left(x + n\varphi_1\right) - \tfrac{1}{4} r_2^n e^{ax} \cos\left(3x + n\varphi_2\right)$$

$$y_n = \tfrac{1}{4} r_1^n 2^{1x} \cos\left(x + n\varphi_1\right) - \tfrac{1}{4} r_2^n 2^{1x} \cos\left(3x + n\varphi_2\right)$$

$$r_1 = \sqrt{(\log 2)^2} + 1 \qquad r_2 = \sqrt{(\log 2)^2} + 3^2$$

$\varphi_1 = \tan^{-1}\left[\frac{1}{\log 2}\right]$ $\varphi_2 = \tan^{-1}\left[\frac{3}{\log 2}\right]$

Q1]e) Express the matrix as the sum of symmetric and skew symmetric matrices.

Solution:- ***(4)***

$$A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A+A') = \frac{1}{2}\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A-A') = \frac{1}{2}\begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 2 & -4 \\ 5 & 6 & 7 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$$

Let $P = \frac{1}{2}(A+A') = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$

$P' = \begin{bmatrix} 1 & -1 & 4 & 7/2 \\ -1 & 1 & 4 & -3/2 \\ 4 & 4 & 7 & 3/2 \\ 7/2 & -3/2 & 3/2 & 0 \end{bmatrix}$

Hence P = P'. P is a symmetric matrix.

Let $Q = \frac{1}{2}(A-A') = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$

$$Q' = \begin{bmatrix} 0 & 1 & 1 & -1/2 \\ -1 & 0 & 2 & 5/2 \\ -1 & -2 & 0 & -1/2 \\ 1/2 & -5/2 & 1/2 & 0 \end{bmatrix}$$

Hence Q = Q'. Q is a skew symmetric matrix.

Q1] f) Evaluate $\lim_{x\to 0} \frac{e^{2x} - (1+x)^2}{x\log(1+x)}$

(4)

Solution :- $\lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{\frac{x\log(1+x)}{x}.x} = \lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{x.x}$

$= \lim_{x\to 0} 1. \frac{e^{2x} - (1+x)^2}{x^2}$

Applying L-Hospital rule

$\lim_{x\to 0} 1. \frac{2e^{2x} - 2(1+x)^1}{2x} = \lim_{x\to 0} 1. \frac{4e^{2x} - 2}{2} = \frac{4e^0 - 2}{2} = 1.$

Q2]a) Show that the roots of x^5=1 can be written as 1,α^1,α^2,α^3,α^4 . hence show that (1-α^1)(1-α^2)(1-α^3)(1-α^4) = 5. ***(6)***

Solution:- x^5 *= 1 = cos0 + isin0*

$\therefore$ x^5 = *cos(2k*π*) + isin(2 k*π*)*

$\therefore$ $x^1 = (\cos(2k\pi) + i\sin(2k\pi))^{\frac{1}{5}}$ = *cos*$\frac{2k\pi}{5}$ + *isin*$\frac{2k\pi}{5}$

Putting k =0,1,2,3,4 we get the five roots as

x_0 *= cos0 + isin0 = 1* , x_1 = *cos*$\frac{2\pi}{5}$ + *isin*$\frac{2\pi}{5}$, x_2 = *cos*$\frac{4\pi}{5}$ + *isin*$\frac{4\pi}{5}$,

x_3 = *cos*$\frac{6\pi}{5}$ + *isin*$\frac{6\pi}{5}$, x_4 = *cos*$\frac{8\pi}{5}$ + *isin*$\frac{8\pi}{5}$.

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$ *we see that* $x_2 = \alpha^2, \ x_3 = \alpha^3, \ x_4 = \alpha^4$

$\therefore$ *the roots are* $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ *and hence*

$\therefore \ x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore \frac{x^5-1}{(x-1)} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x^1 + 1.$

Putting x=1, we get

$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$

Q2]b) Reduce the following matrix to its normal form and hence find its rank.

Solution:- ***(6)***

$$A = \begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$2R_3 - R_1$ $\quad R_1 - R_3$ $\quad R_1 - R_2$ $\quad R_2 - R_4$

$$\begin{bmatrix} 1 & 2 & 6 & -3 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 6 & -3 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 2 & 2 & 7 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 4 & 9 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$R_2 - R_4$ $\quad 4R_2 - R_3$ $\quad R_1 + 2R_4$ $\quad R_3/(-9)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 4 & 9 & -5 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & -10 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -9 & 29 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -9 & 29 \\ 0 & 0 & -2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & -2 & 5 \end{bmatrix}$$

$2R_3 + R_4$ $\quad 6C_2 - C_4$ $\quad 29/9C_3 + C_4$ $\quad R_4 \ /(-13/9)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -29/9 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -13/9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence the given matrix is converted to its normal form

Q2] c) Solve the following equation by Gauss-Seidel method up to four iterations

4x-2y-z = 40, x-6y+2y = -28, x-2y+12z = -86. ***(8)***

Solution:- *we first write the equation as*

$x = \frac{1}{4}[40+2y+z]$ ....(1)

$y = \frac{1}{6}[28+x+2z]$ ....(2)

$z = \frac{1}{12}[-86-x+2y]$ ....(3)

(i) *FIRST ITERATION :-*

we start with the approximation y=0 , z=0 and then we get from (1),

$\therefore x_1 = \frac{1}{4}(40) = 10$

We use this approximation to find y i.e. put x=0, z=0 in (2)

$\therefore y_1 = \frac{1}{6}[28+10+2(0)] = 6.3333$

We use these values of x_1 *and* y_1 *to find* z_1 *i.e. we put x =4 , y = 6.3333 in (3),*

$\therefore z_1 = \frac{1}{12}[-86-10+2(6.3333)] = -6.944$

(ii) *SECOND ITERATION :-*

We use latest values of y and z to find x i.e. we put y =6.3333 , z = -6.9444 in (1)

$\therefore x_2 = \frac{1}{4}[40+2(6.3333)-6.9444] = 11.4306$

We use this approximation to find y i.e. put x=11.4306, z=-6.9444 in (2)

$\therefore y_2 = \frac{1}{6}[28+11.4306+2(-6.9444)] = 4.2569$

i.e. we put x =11.4306 , y = 4.2569 in (3),

$\therefore z_1 = \frac{1}{12}[-86-11.4306+2(4.2569)] = -7.4097$

(iii) *THIRD ITERATION :-*

We use latest values of y and z to find x i.e. we put y =4.2569 , z = -7.40974 in (1)

$\therefore\ x_2 \ = \ \frac{1}{4}[40+2(4.2569)-7.4097] \ = \ 10.2760$

We use this approximation to find y i.e. put x=10.2760 , z = -7.4097 in (2)

$\therefore\ y_2 \ = \ \frac{1}{6}[28+10.2760+2(-7.4097)] \ = \ 3.9094$

i.e. we put x =10.2760 , y = 3.9094 in (3),

$\therefore\ z_1 \ = \ \frac{1}{12}[-86-10.2760+2(3.9094)] \ = \ -7.3714.$

(iv) FOURTH ITERATION:-

We use latest values of y and z to find x i.e. we put y =3.9094 , z = -7.3714 in (1)

$\therefore\ x_2 \ = \ \frac{1}{4}[40+2(3.9094)-7.3714] \ = \ 10.1118$

We use this approximation to find y i.e. put x=10.1118 , z = -7.3714 in (2)

$\therefore\ y_2 \ = \ \frac{1}{6}[28+10.1118+2(-7.3714)] \ = \ 3.8948$

i.e. we put x =10.1118 , y = 3.8448 in (3),

$\therefore\ z_1 \ = \ \frac{1}{12}[-86-10.1118+2(3.8948)] \ = \ -7.3602.$

Hence , upto two places of decimals

x = 10.11 , y = 3.89, z = -7.36.

Q3] a) Investigate for what values of μ and λ the equations x+ y+ z = 6, x + 2y + 3z = 10, x + 2y + λz = μ has

1) No solution

2) A unique solution

3) Infinite number of solutions. *(6)*

Solution:- *we have* $\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\10\\\mu\end{bmatrix}$

By $R_2 - R_1,\ R_3 - R_2$

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & \lambda-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\\mu-10\end{bmatrix}$$

i) *The system has unique solution if the coefficient matrix is non-singular (or the rank A, r= the number of unknowns , n =3).*

This requires $\lambda - 3$ *not equal to 0,*

Hence λ *is not equal to 3.*

Hence the system has unique solution.

ii) *If* $\lambda = 3$ *the coefficient matrix and the augmented matrix becomes*

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \text{ and } \begin{bmatrix}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 0 & 0 & \mu\text{-}10\end{bmatrix}$$

The rank of A = 2 the rank of [A,B] will be also 2 if μ *= 10.*

Thus if $\lambda = 3$ *and* μ *= 10 the system is consistent. But the rank of A (= 2) is less than the number of unknowns (=3). Hence the equation will posses infinite solutions.*

iii) *If* $\lambda = 3$ *and* $\mu \neq 10$ *, the rank of A=2, and the rank of [A,B] = 3. They are not equal and the equations will be inconsistent and will not posses any solution.*

Q3]b) If u = $x^2 + y^2 + z^2$ ***where*** $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

Prove that $\frac{du}{dt} = 4e^{2t}$. ***(6)***

Solution:-

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$\frac{\partial u}{\partial x} = 2x \ , \ \frac{\partial u}{\partial y} = 2y \ , \ \frac{\partial u}{\partial z} = 2z$$

$$\frac{dx}{dt} = e^t \ , \ \frac{dy}{dt} = e^t(\sin t+\cos t) \ , \ \frac{dz}{dt} = e^t(-\sin t+\cos t)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$= 2x(e^t) + 2y(e^t(\sin t+\cos t)) + 2z(e^t(-\sin t+\cos t))$$

$$= 2x(x) + 2y(y+z) + 2z(z-y)$$

$$= 2x^2 + 2y^2 + 2z^2 + 2xy - 2xy$$

$$= 2x^2 + 2y^2 + 2z^2$$

$$= 2(x^2 + y^2 + z^2)$$

$= 2u$ ....(1)

$u = x^2 + y^2 + z^2 = [(e)^t]^2 + (e^t \sin t)^2 + (e^t \cos t)^2$

$= e^{2t} + e^{2t}(\sin^2 t + \cos^2 t)$

$= e^{2t} + e^{2t} = 2e^{2t}$

Substituting value of u in equation (1)

$$\frac{du}{dt} = 2u = 2(2e^{2t}) = 4e^{2t}$$

Hence proved

$$\frac{Du}{dt} = 4e^{2t}.$$

Q3]c) i) Show that $\sin(e^x-1) = x^1 + \frac{x^2}{2} - \frac{5x^4}{24} + ...$ ***(4)***

Solution :- *We have* $\sin(e^x-1) = \sin\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} ....-1\right)$

$$\therefore \sin(e^x-1) = \sin\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} ....\right)$$

But $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - .....$

$$\therefore \sin(e^x-1) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} +....... - \frac{1}{6}\left(x+\frac{x^2}{2}+.......\right)^3 +........$$

$$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} +....... - \frac{x^3}{6} - \frac{x^4}{4} + ...$$

$$= x + \frac{x^2}{2} - \frac{5x^4}{24} + ...$$

Q3]c) ii) Expand $2x^3 + 7x^2 + x - 6$ ***in powers of (x-2)*** ***(4)***

Solution :- *Let* $f(x) = 2x^3 + 7x^2 + x - 6$ *and* $a=2$

$\therefore f'(x) = 6x^2 + 14x + 1$, $f''(x) = 12x + 14$, $f'''(x) = 12$

$\therefore f(2) = 45$, $f'(2) = 53$, $f''(2) = 38$, $f'''(2) = 12$.

Now , f(x) = f(a) + f(x-a)f'(a) + $\frac{(x-a)^2}{2!}$*f''(a) +……………………*

∴ f(x) = f(2) + (x-2)f'(2) + $\frac{(x-2)^2}{2!}$*f''(2) +………………..*

$2x^3 + 7x^2 + x - 6 = 45 + (x-2).53 + (x-2)^{2.19} + (x-2)^{3.2}$

Q4] a) If $x = u+v+w$, $y = uv+vw+uw$, $z = uvw$ and φ is a function of x,y and z.

Prove that $x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z} = u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w}$ **(6)**

Solution:- *φ is a function of x,y and z and x,y,z are themselves functions of u,v,w.*

$$\therefore \frac{\partial\varphi}{\partial u} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial u} = \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(v+w) + \frac{\partial\varphi}{\partial z}vw$$

$$\text{And } \frac{\partial\varphi}{\partial v} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial v} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial v}$$

$$= \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(u+w) + \frac{\partial\varphi}{\partial z}.uw$$

$$\text{And } \frac{\partial\varphi}{\partial w} = \frac{\partial\varphi}{\partial x}.\frac{\partial x}{\partial w} + \frac{\partial\varphi}{\partial y}.\frac{\partial y}{\partial w} + \frac{\partial\varphi}{\partial z}.\frac{\partial z}{\partial w}$$

$$= \frac{\partial\varphi}{\partial x}.1 + \frac{\partial\varphi}{\partial y}(v+u) + \frac{\partial\varphi}{\partial z}.uv$$

Multiplying (1) by u , (2) by v, (3) by w and add

$$\therefore u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w} = (u+v+w)\frac{\partial\varphi}{\partial x} + [(uv+uw) + (vu+vw)+(wv + wu)]\frac{\partial\varphi}{\partial y} + 3uvw\frac{\partial\varphi}{\partial z}$$

$$= (u+v+w)\frac{\partial\varphi}{\partial x} + [2(uv+vw+uw)]\frac{\partial\varphi}{\partial y} + 3uvw\frac{\partial\varphi}{\partial z}$$

$$= x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z}$$

$$\therefore x\frac{\partial\varphi}{\partial x} + 2y\frac{\partial\varphi}{\partial y} + 3z\frac{\partial\varphi}{\partial z} = u\frac{\partial\varphi}{\partial u} + v\frac{\partial\varphi}{\partial v} + w\frac{\partial\varphi}{\partial w}$$

Q4] b) If $\tan(\theta + i\varphi) = \tan\alpha + i\sec\alpha$

Prove that 1) $e^{2\varphi} = \cot\frac{\alpha}{2}$ **2)** $2\theta = m\pi + \frac{\pi}{2} + \alpha.$ **(6)**

Solution :- $\tan(\theta + i\varphi) = \tan\alpha + i\sec\alpha$ $\quad\therefore$ $\tan(\theta - i\varphi) = \tan\alpha - i\sec\alpha$

$$\therefore \tan 2\theta = \tan[(\theta + i\varphi) + (\theta - i\varphi)] = \frac{\tan(\theta+i\varphi) + \tan(\theta-i\varphi)}{1-\tan(\theta+i\varphi)\tan(\theta-i\varphi)}$$

$$= \frac{\tan(\theta+i\sec\alpha) + \tan(\theta-i\sec\alpha)}{1-\tan(\theta+i\sec\alpha)\tan(\theta-i\sec\alpha)} = \frac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)} = \frac{2\tan\alpha}{-2\tan^2\alpha} - \cot\alpha =$$

$$\tan\left(\frac{\pi}{2} + \alpha\right)$$

$$\therefore 2\theta = n\pi + \frac{\pi}{2} + \alpha. \quad \text{(general value).}$$

Again $\tan(2i\varphi) = \tan[(\theta + i\varphi) - (\theta - i\varphi)]$

$$= \frac{\tan(\theta+i\sec\alpha) - \tan(\theta-i\sec\alpha)}{1+\tan(\theta+i\sec\alpha)\tan(\theta-i\sec\alpha)}$$

$$\therefore i\tanh 2\varphi = \frac{2i\sec\alpha}{2\sec^2\alpha} = i\cos\alpha \quad \therefore \quad \tanh 2\varphi = \cos\alpha$$

$$\therefore 2\varphi = \tan h\hat{}(-1)\,(\cos\alpha(\cos\alpha \;\frac{1}{2}\log\left[\frac{1+\cos\alpha}{1-\cos\alpha}\right] = \frac{1}{2}\log\left[\frac{2\cos^2\left(\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right)}\right] og\cot\frac{\alpha}{2}$$

Q 4]c) Find the roots of the equation $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 2.1 correct to 3 places of decimals using Regula Falsi method.

Solution:- **(8)**

Given that a=2 and b =2.1.

$$f(2) = (2)^4 + (2)^3 - 7(2)^2 - 2 + 5 = -1.$$

$$f(2.1) = (2.1)^4 + (2.1)^3 - 7(2.1)^2 - (2.1) + 5 = 0.739100.$$

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} = (2\times0.73910—1))\times(2.1)0.739100—1 = 2.05750. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$f((x_1) = (2.05750)^4 + (2.05750)^3 - 7(2.05750)^2 - 2.05750 + 5$$

$$= -0.05973.$$

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1)-f(a)} = (2\times(-0.05973)—1))\times(2.05750)0.05973—1 = 2.061152$$

$\ldots\ldots\ldots\ldots\ldots(2)$

$f(x_2)$ = $(2.061152)^4 + (2.061152)^3 - 7(2.061152)^2 - 2.061152 + 5$

= 0.005326.

$x_3 = \frac{af(x_2) - x_2 f(a)}{f(x_2) - f(a)}$ = (2×(0.005326)—1))×(2.061152)0.005326—1 = 2.06082.

……………(3)

$f(x_3)$ = $(2.06082)^4 + (2.06082)^3 - 7(2.06082)^2 - 2.06082 + 5$

= -0.000582.

$x_4 = \frac{af(x_3) - x_3 f(a)}{f(x_3) - f(a)}$ = (2×(- 0.000582)—1))×(2.06082)0.000582—1 = 2.0608.

……………(4)

Hence from (4) and (3) iteration we get that value of x is coinciding.

Therefore the final value of x is 2.0608.

Q5] a) If $y = (x + \sqrt{x^2 - 1})^m$, Prove that

$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$ (6)

Solution:-

$y = (x + \sqrt{x^2 - 1})^m$

taking + sign before the radical

$\therefore y_1 = m[(x + \sqrt{x^2 - 1})^{m-1}].[1 + \frac{x}{\sqrt{x^2-1}}]$

$= m(x + \sqrt{x^2 - 1})^m). \frac{x}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$

$\sqrt{x^2 - 1} . y_1 = my$

Differentiating again w.r.t x,

$\sqrt{x^2 - 1} . y_2 + \frac{x}{\sqrt{x^2-1}} y_1 = m y_1$

$(x^2 - 1) y_2 + xy_1 = m\sqrt{x^2 - 1}. y_1 = m.my = my^2$

$(x^2 - 1) y_2 + xy_1 - my^2 = 0$

Hence after applying lebnitz's theorem we get,

$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Q5]b) Using the encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ encode and decode the message I*LOVE*MUMBAI.

Solution:-

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

*	Y.	Z
27.	25.	26

I	*	L	O	V	E	*	M	U	M	B	A	I	*
9	27	12	15	22	5	27	13	21	13	2	1	9	27

Encoding the message includes the following process.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9 & 12 & 22 & 27 & 12 & 2 & 9 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 9+27 & 12+15 & 22+5 & 27+13 & 12+13 & 2+1 & 9+27 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 27 & 27 & 40 & 25 & 3 & 36 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$$

Hence the encoded message we get as ,

36, 27, 27, 15, 27, 5, 40, 13, 25, 13, 3, 1, 36, 27.

Now the process of decoding is as follows.

Inverse of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *is* $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 36 & 27 & 27 & 40 & 25 & 3 & 36 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 36-27 & 27-15 & 27-5 & 40-13 & 25-13 & 3-1 & 36-27 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$$

$= \begin{bmatrix} 9 & 12 & 22 & 27 & 12 & 2 & 9 \\ 27 & 15 & 5 & 13 & 13 & 1 & 27 \end{bmatrix}$

Hence the original message is obtained again after encoding and decoding.

Q5]c) i) Considering only principal values separate into real and imaginary parts

$i^{\log(i+1)}$. **(4)**

Solution:- *let* $Z = i^{\log(i+1)}$ $\therefore$ *logZ = log(1+i).logi*

But $\log(i+1) = \log\sqrt{2} + i\tan^{-1} 1 = \log\sqrt{2} + i\frac{\pi}{4}$ *and* $\log i = i.\frac{\pi}{2}$

$\therefore \log Z = (\log\sqrt{2} + i\frac{\pi}{4}).\, i.\frac{\pi}{2} = [\frac{1}{2}\log 2 + i\frac{\pi}{4}\pi)/2 = \frac{-\pi^2}{8} + i\frac{\pi}{4}\log 2 = e^{-\frac{\pi^2}{8}+i\theta} =$

$e^{-\frac{\pi^2}{8}.i\theta}$ *where* $\theta = \frac{\pi}{4}\log 2 = e^{-\frac{\pi^2}{8}}[\cos\theta + i\sin\theta]$

$\therefore$ *Real part of Z* $= e^{-\frac{\pi^2}{8}}\cos\theta = e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$ $\therefore$*nary part of Z* $= e^{-\frac{\pi^2}{8}}\sin\left(\frac{\pi}{4}\log 2\right)$

Q5]c) ii) Show that $i\log(\frac{x-i}{x+i}) = \pi - 2\tan^{-1} x$ **(4)**

Solution:- *we have* $\log(x+i) = \frac{1}{2}\log(x^2+1) + i\tan^{-1}\frac{1}{x}$

and $\log(x-i) = \frac{1}{2}\log(x^2+1) - i\tan^{-1}\frac{1}{x}$

$\log(\frac{x-i}{x+i}) = \log(x-i) - \log(x+i)$

$= -2i\tan^{-1}\frac{1}{x} = -2i(\frac{\pi}{2} - \tan^{-1} x)$

$\therefore \log(\frac{x-i}{x+i}) = -i(\pi - 2\tan^{-1} x)$

$\therefore i\log(\frac{x-i}{x+i}) = (\pi - 2\tan^{-1} x)$.

Q6]a) Using De Moivre's theorem prove that

$\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)$ *(6)*

Solution:- *Let as above* $x = \cos\theta + i\sin\theta$, *then* $\frac{1}{x} = \cos\theta - i\sin\theta$

$(2\cos\theta)^6 = (x + \frac{1}{x})^6$

$= x^6 + 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} + 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} + 6x^1.\frac{1}{x^5} + \frac{1}{x^6}$

$= x^6 + 6x^5 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}$ ……………………(1)

$(2i\sin\theta)^6 = (x - \frac{1}{x})^6$

$= x^6 - 6x^5 + 15x^2 - 20 + 15\frac{1}{x^2} - 6\frac{1}{x^4} + \frac{1}{x^6}$ ……………………..(2)

$(2\sin\theta)^6 = x^6 + 6x^5 - 15x^2 + 20 - 15\frac{1}{x^2} + 6\frac{1}{x^4} - \frac{1}{x^6}$

Subtracting (2) from (1),

$2^6(\cos^6\theta - \sin^6\theta) = [x^6 + 6x^5 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}] -$

$[-x^6 + 6x^5 - 15x^2 + 20 - 15\frac{1}{x^2} + 6\frac{1}{x^4} - \frac{1}{x^6}]$

$= 2(x^6 + \frac{1}{x^6}) + 15(x^2 + \frac{1}{x^2})$

$= 2\cos 6\theta + 15\cos 2\theta$ …………………[$(x^6 + \frac{1}{x^6}) = \cos 6\theta$]

$\therefore 2^6(\cos^6\theta - \sin^6\theta) = 2\cos 6\theta + 15\cos 2\theta.$

$\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)$

Q6] b) If $u = \sin^{-1}\left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)^{1/2}$ **, Prove that**

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(\tan^2 u + 13) \qquad (6)$$

Solution:-

$Z = \sin u = \sqrt{\left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)} = f(u) = F(X,Y)$ say.

Putting $X = xt$, $Y = yt$

$$F(X,Y) = \sqrt{\left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)} = \sqrt{\left(\frac{(xt)^{\frac{1}{3}} + (yt)^{\frac{1}{3}}}{(xt)^{\frac{1}{2}} - (yt)^{\frac{1}{2}}}\right)} = \sqrt{\frac{t^{1/3}}{t^{1/2}}\left(\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}}\right)} = t^{-1/12} f(x,y)$$

Thus $Z = f(u) = \sin u$ is a homogenous function of x, y of degrees 1/12

Hence, by the above corollary.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

Where, $g(u) = n\frac{f(u)}{f'(u)} = \frac{-1}{12}.\frac{\sin u}{\cos u} = \frac{-1}{12}\tan u$

$$g'(u)-1 = \frac{-1}{12}\sec^2 u - 1 = \frac{-1}{12}(1-\tan^2 u) - 1 = \frac{-1}{12}\tan^2 u - \frac{13}{12}$$

$$= \frac{-1}{12}(\tan^2 u + 13)$$

$$\therefore\ g(u)[g'(u)-1] = \left(\frac{-1}{12}\tan u\right)\left(\frac{-1}{12}(\tan^2 u + 13)\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(\tan^2 u + 13)$$

Q6]c) Find the maxima and minima of $x^3y^2(1-x-y)$ **(8)**

Solution :- we have $f(x) = x^3y^2(1-x-y)$

Step 1 :- $f_x = y^2[3x^2(1-x-y) - x^3] = y^2(3x^2-4x^3-3x^2y)$

$= (3x^2y^2-4x^3y^2-3x^2y^3)$

$f_y \quad = x^3\,[2y(1-x-y)y(1-x-y)y^2] \quad = \quad x^3(2y-2xy-3y^2)$

$= (2yx^3 - 2x^4y - 3x^3y^2)$

$\therefore \quad f_{xx} = 6y^2x^1 - 12x^2y^2 - 6x^1y^3$

$\therefore \quad f_{xy} = 6y^1x^2 - 8x^3y^1 - 9x^2y^2$

$\therefore \quad f_{yy} = 2x^3 - 2x^4 - 6x^3$

Step 2:- we now solve for $f_y = 0$, $f_x = 0$

$\therefore 3y^2x^2 - 4x^3y^2 - 3x^2y^3 = 0$ *i.e.* $y^2x^2(3-4x-3y) = 0$

And $2y^1x^3 - 2x^4y^1 - 3x^3y^2 = 0$ *i.e.* $y^1x^3(2-2x-3y) = 0$

$\therefore x = 0, y = 0$ *and* $(3-4x-3y) = 0,\ 2-2x-3y = 0$

Subtracting we get $1-2x = 0$

$\therefore x = ½ \quad \therefore 3y = 3-4(1/2) = 1 \quad \therefore y = \frac{1}{3}$

$\therefore$ *(0,0) and (1/2 , 1/3) are stationary points.*

Step 3:- at $x = 0, \ y = 0, r = 0, s = 0, t = 0 \quad \therefore rt - s^2 = 0$

At $x = ½ \ , \ y = 1/3$

$r = f_{xx} = 6(1/2)(1/9) - 12(1/4)(1/9) - 6(1/2)(1/27))(1/27)\frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9}$

$s = f_{xy} = 6(1/4)(1/3) - 8(1/8)(1/3) - 9(1/4)(1/9)9) \quad = ½ - \frac{1}{3} - ¼ = -\frac{1}{12}$

$t = f_{xy} = 2(1/8) - 2(1/16) - 6(1/8)(1/3) \quad = ¼ - \frac{1}{8} - ¼ = -\frac{1}{8}$

$\therefore rt - s^2 = (-1/9)(-1/8) - (1/12)(1/12) \quad = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$

And $r = -1/9 < 0 \quad \therefore$ *f(x,y) is a maxima*

Maximum value $= \frac{1}{8}.\frac{1}{9}\left(1-\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{432}$
